Integrable systems related to separation of variables on symmetric spaces of rank 1

Holger Dullin

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Separation of variables gives a way to construct families of integrable systems. We study such (quantum) integrable systems on spheres and other symmetric spaces of rank 1. For example choosing a separating coordinate system for the geodesic flow on S^3 induces a Liouville integrable system, which after symplectic quotient by the S^1 action induced by the geodesic flow induces an integrable system on the Grassmannian $Gr(2,4)$ which is $S^2 \times S^2$. We show that in this family there is only one such integrable system on $Gr(2,n+1)$ that is toric. The image of the classical momentum map on S^3 is a cone over a square, containing the joint spectrum of the corresponding quantum integrable system. Repeating the construction for $\mathbb{R}P(3) = SO(3)$ also gives a cone, but with half the volume and the square rotated by 45 degrees. The arrangement of the joint spectra is different, but of course such that Weyl's law holds in the semi-classical limit with half the number of states for $SO(3)$. Generalisations to $\mathbb{C}P(n)$ and using a recent result with Bolsinov, Matveev, Nikolayevsky are discussed. This is joint work with Damien McLeod.